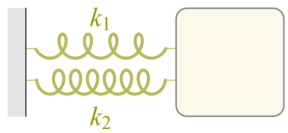
Final Review

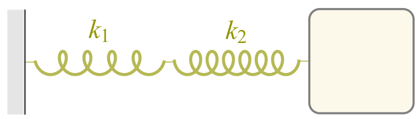
A Puzzle...

Consider two massless springs with spring constants k_1 and k_2 and the same equilibrium length.

1. If these springs act on a mass m in parallel, they would be equivalent to a single spring with spring constant k_{parallel}. Find k_{parallel}.



2. If these springs act on a mass m in series, they would be equivalent to a single spring with spring constant k_{series}. Find k_{series}.



<u>Solution</u>
1. For a displacement x from equilibrium, the force felt by the mass will be $-(k_1 + k_2)\,x$, and therefore $k_{\text{parallel}} = k_1 + k_2$.
2. Consider a massless object between the first and second spring. When the full system is displaced by a distance x, the first spring will be stretched by x_1 and the second spring by x_2 so that the total displacement is given by $x_1 + x_2 = x$ and there is no net force on the massless object $k_1\,x_1 = k_2\,x_2$. This implies

$$x_1 = \frac{k_2}{k_1 + k_2}\,x \qquad (1)$$

$$x_2 = \frac{k_1}{k_1 + k_2}\,x \qquad (2)$$

Therefore, the net force felt on the mass m equals

$$k_2\,x_2 = \frac{k_1\,k_2}{k_1 + k_2}\,x \equiv k_{\text{series}}\,x \qquad (3)$$

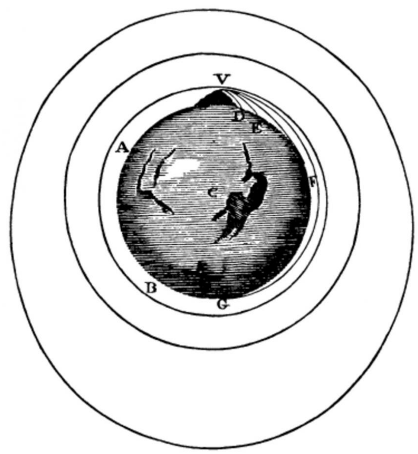
where we have defined the effective spring constant k_{series} of the system. Note that

$$\frac{1}{k_{\text{series}}} = \frac{k_1 + k_2}{k_1\,k_2} = \frac{1}{k_1} + \frac{1}{k_2} \qquad (4)$$

which is the called the *<u>harmonic sum</u>* of k_1 and k_2. □

Superman Orbits

<u>Example</u>
Superman stands on top of a mountain and throws a series of stone horizontally, with each subsequent stone thrown at an ever greater velocity. Describe the motions of the stones as the initial velocity of the throws increase.



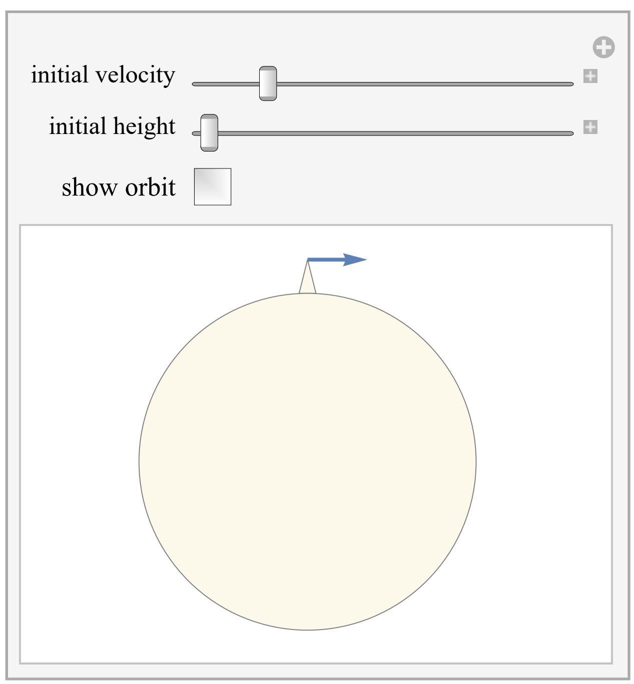
Solution

From Kepler's First Law, the stones travel in an ellipse with the Earth's center as one of its foci.

Let us orient our axes so that $+\hat{x}$ points to the right and $+\hat{y}$ points up away from the center of the Earth. Given Kepler's First Law, the rock must travel in an ellipse. Because the initial velocity is directly horizontal - perpendicu lar to the line between yourself and the center of the Earth - this implies that:

1. The starting point of the rock will be the highest point
2. The semi-major and semi-minor axes of the ellipse must be along the $\hat{y}$ and $\hat{x}$ directions, respectively
3. The orbit of the particle must be symmetric in the x-direction

Here we see what the orbits can look like, with the center of the ellipse shown in orange and the two foci shown in dark green:



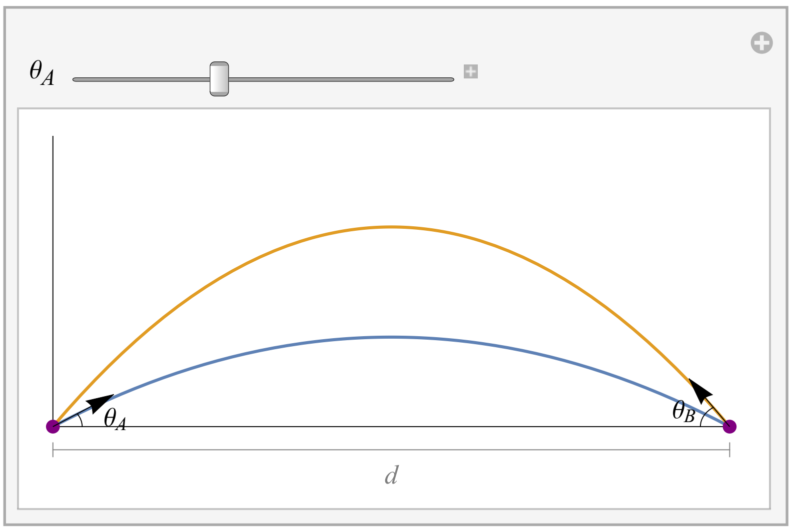
If we look closely at Newton's drawing, we see that it does look correct. However, note that:

1. If the rock travels $\frac{1}{4}$ of the way around the world, its final velocity as it crashes to the ground will not be parallel

to the ground but instead will point towards the Earth
2. If the rock travels more than $\frac{1}{2}$ way around the world, then it must be in an orbit and will not reach land until it returns to its starting position. In particular, a rock cannot be thrown $\frac{3}{4}$ of the way around the world! □

Clown Cannon

Example
During an incredible *Cirque Du Soleil* performance, two cannons A and B separated by a distance d simultaneously fire clowns into the air. Both clowns are launched at velocity v but at different angles θ_A and θ_B so that the two clown high-five in midair and then land in the other cannon!
1. Assume θ_B is fixed. Find θ_A in terms of θ_B so that the two clowns cover the same distance d but do not collide in midair
2. Will both clowns reach the maximum height of their trajectory at the same time?
3. What is the minimum distance between the two clowns during their trajectory? In other words, how far do the clowns have to reach to high five each other? (Check your answer in the limits $\theta_A = 0, \frac{\pi}{4}$, and $\frac{\pi}{2}$)



Solutions
1. Both clowns have acceleration g downwards. Integrating this, we find the velocities

$$\vec{v}_A[t] = v \operatorname{Cos}[\theta_A]\,\hat{x} + (v \operatorname{Sin}[\theta_A] - g\,t)\,\hat{y} \tag{5}$$
$$\vec{v}_B[t] = -v \operatorname{Cos}[\theta_B]\,\hat{x} + (v \operatorname{Sin}[\theta_B] - g\,t)\,\hat{y} \tag{6}$$

(As a quick aside, these two equations enable us to answer Part 2 of the question: Will the clowns both reach their maximum heights at the same time? Absolutely not, since the maximum height is determined by $v \operatorname{Sin}[\theta] - g\, t_{\text{max height}} = 0$, and $\operatorname{Sin}[\theta]$ will be different for the two cannons.)

We now integrate the velocity to obtain the positions of the cupcakes. Cupcake A started at $(0, 0)$ and cupcake B started at $(d, 0)$. Thus, we can integrate the above equations using $\vec{v} = \frac{d\vec{r}}{dt}$ to obtain

$$\vec{r}_A[t] = v\,t \operatorname{Cos}[\theta_A]\,\hat{x} + \left(v\,t \operatorname{Sin}[\theta_A] - \frac{1}{2} g\,t^2\right)\hat{y} \tag{7}$$
$$\vec{r}_B[t] = (d - v\,t \operatorname{Cos}[\theta_B])\,\hat{x} + \left(v\,t \operatorname{Sin}[\theta_B] - \frac{1}{2} g\,t^2\right)\hat{y} \tag{8}$$

Cupcake A hits the ground when its y-component is zero,

$$0 = v\,t\,\mathrm{Sin}[\theta_A] - \frac{1}{2}\,g\,t^2 \tag{9}$$

which solves to $t = 0$ or $t = \frac{2\,v}{g}\,\mathrm{Sin}[\theta_A]$. We want the second solution, which allows us to solve for d,

$$d = v\left(\frac{2\,v}{g}\,\mathrm{Sin}[\theta_A]\right)\mathrm{Cos}[\theta_A] = \frac{v^2}{g}\,\mathrm{Sin}[2\,\theta_A] \tag{10}$$

Similarly solving when the y-component of $\vec{r}_B$ equals 0 allows us to find when cupcake B hits the ground, which by the symmetry of the equations must equal $t = \frac{2\,v}{g}\,\mathrm{Sin}[\theta_B]$. Since cupcake B hits cannon A,

$$0 = d - v\left(\frac{2\,v}{g}\,\mathrm{Sin}[\theta_B]\right)\mathrm{Cos}[\theta_B] \tag{11}$$

which simplifies to

$$d = \frac{v^2}{g}\,\mathrm{Sin}[2\,\theta_B] \tag{12}$$

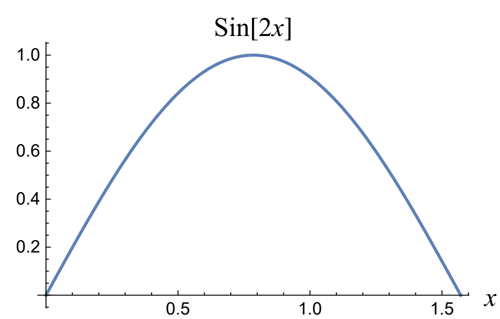
Substituting back into Equation (10),

$$\frac{v^2}{g}\,\mathrm{Sin}[2\,\theta_A] = \frac{v^2}{g}\,\mathrm{Sin}[2\,\theta_B] \tag{13}$$

or equivalently

$$\mathrm{Sin}[2\,\theta_A] = \mathrm{Sin}[2\,\theta_B] \tag{14}$$

We would like to solve for θ_B in terms of θ_A. One obvious solution is to take ArcSin of both sides of the equation and obtain $\theta_B = \theta_A$, but we specifically don't want this solution (since we don't want the cupcakes to hit each other). So what does this mean? At this point, it helps to visualize $\mathrm{Sin}[2\,x]$,



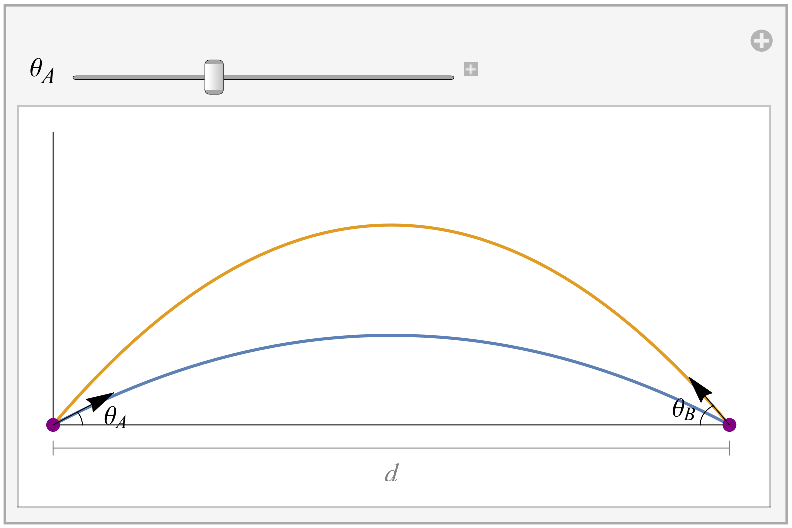
The answer lies in the fact that $\mathrm{Sin}[2\,x] = \mathrm{Sin}\left[2\left(\frac{\pi}{2} - x\right)\right]$. Therefore we can write the above relation for θ_A and θ_B as

$$\mathrm{Sin}[2\,\theta_A] = \mathrm{Sin}\left[2\left(\frac{\pi}{2} - \theta_B\right)\right] \tag{15}$$

and upon taking the ArcSin of both sides,

$$\theta_B = \frac{\pi}{2} - \theta_A \tag{16}$$

Note that when $\theta_A = \frac{\pi}{4}$, $\theta_B = \frac{\pi}{4}$ which is the only time that both cupcakes are forced onto the same trajectory.



2. As discussed above, the two clowns will not reach their maximum heights at the same time. The clown that goes higher will be in the air for longer, but the horizontal velocity is constant during projectile motion, so the clown that takes the high road will reach their maximum height later than the clown that takes the low road, just like in the Loch Lomond song.

3. Substituting Equation (16) into Equations (7) and (8) and simplifying using trig identities,

$$\vec{r}_A = v\,t\,\text{Cos}[\theta_A]\,\hat{x} + \left(v\,t\,\text{Sin}[\theta_A] - \frac{1}{2}\,g\,t^2\right)\hat{y} \tag{17}$$

$$\vec{r}_B = (d - v\,t\,\text{Sin}[\theta_A])\,\hat{x} + \left(v\,t\,\text{Cos}[\theta_A] - \frac{1}{2}\,g\,t^2\right)\hat{y} \tag{18}$$

We would like to minimize the distance between the two cupcakes over t. This distance equals

$$|\vec{r}_A - \vec{r}_B| = \sqrt{\left(\{v\,t\,(\text{Cos}[\theta_A] + \text{Sin}[\theta_A]) - d\}^2 + \{v\,t\,(\text{Sin}[\theta_A] - \text{Cos}[\theta_A])\}^2\right)} \tag{19}$$

To find the minima, we are going to need to differentiate this function over t and set it equal to 0. However, taking the derivative of a square root is incredibly messy. To simplify the calculation, notice that the minima of $|\vec{r}_A - \vec{r}_B|$ occurs at the same time as the minima of $|\vec{r}_A - \vec{r}_B|^2$, so instead we will differentiate the *significantly* simpler function

$$|\vec{r}_A - \vec{r}_B|^2 = \{v\,t\,(\text{Cos}[\theta_A] + \text{Sin}[\theta_A]) - d\}^2 + \{v\,t\,(\text{Sin}[\theta_A] - \text{Cos}[\theta_A])\}^2 \tag{20}$$

Differentiating with respect to t,

$$\begin{aligned}\frac{d|\vec{r}_A - \vec{r}_B|^2}{dt} &= 2\,\{v\,t\,(\text{Cos}[\theta_A] + \text{Sin}[\theta_A]) - d\}\,\{v\,(\text{Cos}[\theta_A] + \text{Sin}[\theta_A])\}\\ &\quad + 2\,\{v\,t\,(\text{Sin}[\theta_A] - \text{Cos}[\theta_A])\}\,\{v\,(\text{Sin}[\theta_A] - \text{Cos}[\theta_A])\}\end{aligned} \tag{21}$$

Simplifying and setting $0 = \frac{d|\vec{r}_A - \vec{r}_B|^2}{dt}$,

$$\begin{aligned}0 &= v^2\,t\,(\text{Cos}[\theta_A] + \text{Sin}[\theta_A])^2 + v^2\,t\,(\text{Sin}[\theta_A] - \text{Cos}[\theta_A])^2 - v\,d\,(\text{Cos}[\theta_A] + \text{Sin}[\theta_A])\\ &= 2\,v^2\,t - v\,d\,(\text{Cos}[\theta_A] + \text{Sin}[\theta_A])\end{aligned} \tag{22}$$

which we can solve for the time of minimum distance,

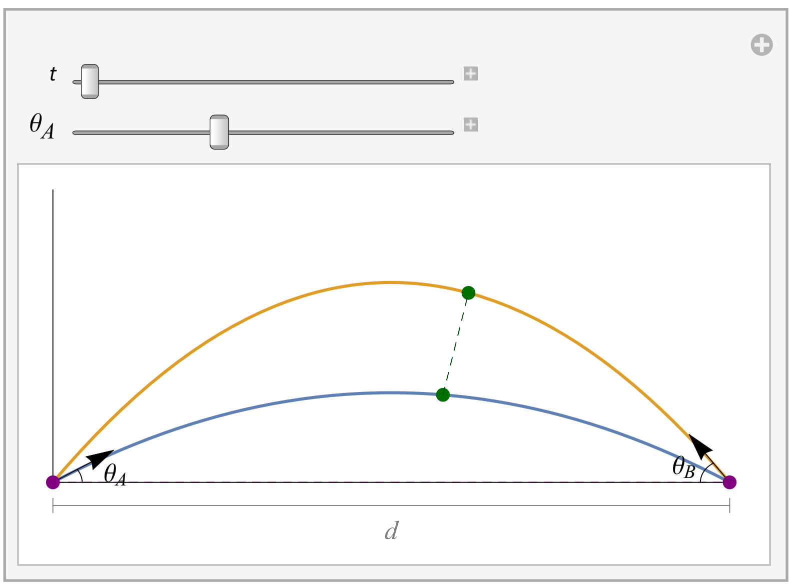
$$t_{\text{min distance}} = \frac{d}{2\,v}\,(\text{Cos}[\theta_A] + \text{Sin}[\theta_A]) \tag{23}$$

To find the minimum distance, we substitute this time back into Equation (19) to obtain

$$
\begin{aligned}
r_{\text{min distance}} &= \sqrt{\{v\, t_{\text{min distance}}\,(\text{Cos}[\theta_A] + \text{Sin}[\theta_A]) - d\}^2 + \{v\, t_{\text{min distance}}\,(\text{Sin}[\theta_A] - \text{Cos}[\theta_A])\}^2} \\
&= \sqrt{\left\{\tfrac{d}{2}\,(1 + \text{Sin}[2\,\theta_A]) - d\right\}^2 + \left\{\tfrac{d}{2}\,\text{Cos}[2\,\theta_A]\right\}^2} \\
&= \tfrac{d}{2}\sqrt{(\text{Sin}[2\,\theta_A] - 1)^2 + \text{Cos}[2\,\theta_A]^2} \\
&= \tfrac{d}{2}\sqrt{2\,(1 - \text{Sin}[2\,\theta_A])} \\
&= \tfrac{d}{\sqrt{2}}\sqrt{(\text{Cos}[\theta_A] - \text{Sin}[\theta_A])^2} \\
&= \tfrac{d}{\sqrt{2}}\,\left|\text{Cos}[\theta_A] - \text{Sin}[\theta_A]\right|
\end{aligned}
$$

That's a pretty neat formula! Note that $t_{\text{min distance}}$ does not occur halfway through the flights of either cupcake A or cupcake B ($t = \frac{v}{g}\,\text{Sin}[\theta_A]$ and $t = \frac{v}{g}\,\text{Sin}[\theta_B]$, respectively) except for the special cases $\theta_A = 0$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$.

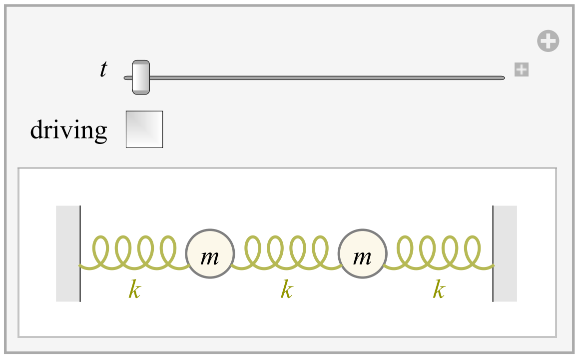
The two cases $\theta_A = 0$ and $\theta_A = \frac{\pi}{2}$ are trivial since $d = 0$ and the two clowns start off at the same spot, so that $r_{\text{min distance}} = 0$. For $\theta_A \approx \frac{\pi}{4}$, both clowns take roughly the same path (and if $\theta_A = \frac{\pi}{4}$ exactly, then both clowns collide in midair), so we expect $r_{\text{min distance}} \approx 0$, as is indeed seen by the formula above. For any other trajectory, the point of closest approach between the two clowns will not be when either cupcake is at the peak of its flight. Here is a diagram of the trajectories of the flying cupcakes (shown in purple), with the points of closest approach shown in green.



Driven Oscillations

Example

Two identical masses are attached to three identical springs as shown below. The system starts off at rest with all of the springs at their unstretched length. At time $t = 0$, the left mass is subjected to a driving force $F_d\,\text{Cos}[2\,\omega\,t]$ and the right mass to a driving force $2\,F_d\,\text{Cos}[2\,\omega\,t]$, where $\omega = \left(\frac{k}{m}\right)^{1/2}$. Our goal will be to find the resulting displacement of the left mass $x_L[t]$ and the right mass $x_R[t]$.



1. Write down the accelerations $\ddot{x}_L[t]$ and $\ddot{x}_R[t]$ of the left and right masses in terms of the forces acting upon each mass. (Check your answer for the special cases $x_L = x_R$, $x_L = 0$, and $x_R = 0$ to ensure you do not have any signs flipped)
2. Add your two equations from Part 1 to find the sum of accelerations $\ddot{x}_L[t] + \ddot{x}_R[t]$ and the difference in accelerations $\ddot{x}_L[t] - \ddot{x}_R[t]$
3. Suppose the solutions for the sum and difference of the displacements have the forms

$$\begin{aligned} x_L[t] + x_R[t] &= A\,\mathrm{Cos}[2\,\omega\,t] \\ x_L[t] - x_R[t] &= B\,\mathrm{Cos}[2\,\omega\,t] \end{aligned} \tag{25}$$

Substitute these equations into the differential equations from Part 2 and determine A and B.
4. Find $x_L[t]$ and $x_R[t]$. Explain the motion of the left mass using force analysis

<u>Solution</u>

1. The equations of motion for the two masses are

$$\begin{aligned} m\,\ddot{x}_L[t] &= -k\,x_L[t] + k\,(x_R[t] - x_L[t]) + F_d\,\mathrm{Cos}[2\,\omega\,t] \\ m\,\ddot{x}_R[t] &= -k\,x_R[t] - k\,(x_R[t] - x_L[t]) + 2\,F_d\,\mathrm{Cos}[2\,\omega\,t] \end{aligned} \tag{26}$$

It always helps to check these equations in various limits. For example, if $x_L[t] = x_R[t]$, the middle spring will not exert a force and the two masses will feel a leftwards force if they are displaced to the right ($0 < x_L[t]$, $x_R[t]$). If $x_L[t] = 0$ and $x_R[t] > 0$, the left mass will feel a spring force to the right while the right mass will feel two spring forces to the left. If $x_R[t] = 0$ and $x_L[t] > 0$, the left mass will feel two spring forces to the left and the right mass will feel a spring force to the right. You can check that all of these statements are supported by the equations above.

2. The sum and difference of the two accelerations is given by

$$\begin{aligned} \ddot{x}_L[t] + \ddot{x}_R[t] &= -\omega^2(x_L[t] + x_R[t]) + 3\,\frac{F_d}{m}\,\mathrm{Cos}[2\,\omega\,t] \\ \ddot{x}_L[t] - \ddot{x}_R[t] &= -3\,\omega^2(x_L[t] - x_R[t]) - \frac{F_d}{m}\,\mathrm{Cos}[2\,\omega\,t] \end{aligned} \tag{27}$$

where we have used $\omega^2 = \frac{k}{m}$. Note that the top differential equation is written purely in terms of $x_L[t] + x_R[t]$ and its second derivative, while the bottom equation is written purely in terms of $x_L[t] - x_R[t]$ and its derivative. Thus, in terms of the sum and difference of the displacements, these two differential equations are uncoupled. Each equation has the same solution as a driven harmonic oscillator.

3. Substituting in the forms of Equation (25) into Equation (27), we obtain

$$\begin{aligned} -4\,\omega^2\,A\,\mathrm{Cos}[2\,\omega\,t] &= -\omega^2\,A\,\mathrm{Cos}[2\,\omega\,t] + 3\,\frac{F_d}{m}\,\mathrm{Cos}[2\,\omega\,t] \\ -4\,\omega^2\,B\,\mathrm{Cos}[2\,\omega\,t] &= -3\,\omega^2\,B\,\mathrm{Cos}[2\,\omega\,t] - \frac{F_d}{m}\,\mathrm{Cos}[2\,\omega\,t] \end{aligned} \tag{28}$$

Note that the $\mathrm{Cos}[2\,\omega\,t]$ terms drop from both equations - this must always happen because A and B cannot have time dependence. Simplify the above equations,

$$\begin{aligned} A &= -\frac{F_d}{k} \\ B &= \frac{F_d}{k} \end{aligned} \tag{29}$$

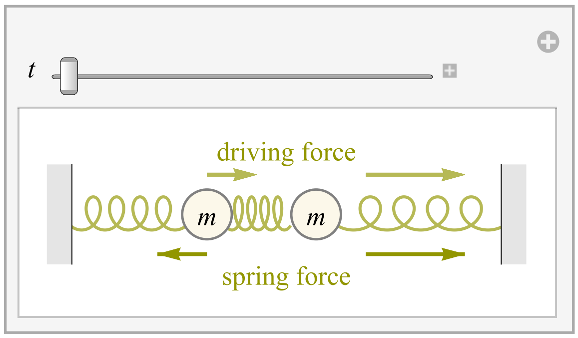
4. Note that

$$\begin{aligned} x_L[t] &= \frac{(x_L[t]+x_R[t])+(x_L[t]-x_R[t])}{2} \\ x_R[t] &= \frac{(x_L[t]+x_R[t])-(x_L[t]-x_R[t])}{2} \end{aligned} \tag{30}$$

Substituting in Equations (25) and (29),

$$\begin{aligned} x_L[t] &= \frac{-\frac{F_d}{k}\,\text{Cos}[2\,\omega\,t]+\frac{F_d}{k}\,\text{Cos}[2\,\omega\,t]}{2} = 0 \\ x_R[t] &= \frac{-\frac{F_d}{k}\,\text{Cos}[2\,\omega\,t]-\frac{F_d}{k}\,\text{Cos}[2\,\omega\,t]}{2} = -\frac{F_d}{k}\,\text{Cos}[2\,\omega\,t] \end{aligned} \tag{31}$$

The motion of the system is shown below.

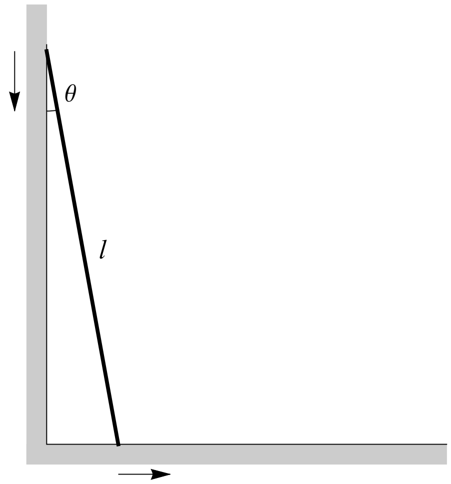


Apparently, the left mass is stationary! This must mean that the net forces on the left mass are always zero, which we can quickly verify. Since $x_L[t] = 0$, the only spring force on the left mass is given by $k\,x_R[t] = -F_d\,\text{Cos}[2\,\omega\,t]$ according to Equation (31). This force is exactly balanced by the driving force exerted on the left mass, thereby resulting in no net force and hence no net motion! □

Leaning Ladder

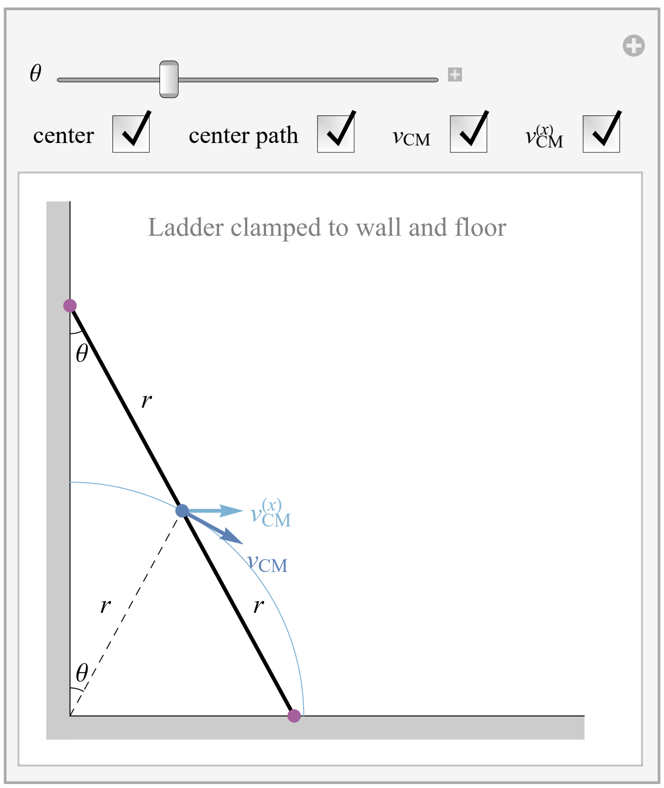
Example

A ladder of length l and uniform mass density stands on a frictionless floor and leans against a frictionless wall at an initial angle θ_0 relative to the vertical. It is released from rest, whereupon the bottom end slides away from the wall, and the top end slides down the wall. When will the ladder lose contact with the wall?



<u>Solution 1</u>

Let $r = \frac{l}{2}$. The ladder has a moment of inertia $\frac{1}{12}\, m\, l^2 = \frac{1}{3}\, m\, r^2$ about its center, but we will use the more general $I = \eta\, m\, r^2$. Let θ be the angle between the wall and the ladder, which also equals the vertical angle from the wall to the center of mass. This symmetry implies that the center of the ladder (while it stays in contact with the wall) travels along a circle! Here is what the motion of the ladder would look like if its ends were forced to remain on the floor and wall (for example, if its ends were clamped onto frictionless rails on the floor and wall).



Before we begin solving this problem, let's gain some insight into the motion of the ladder and understand why it should come away from the wall before it hits the ground. The forces acting on the ladder are the normal forces N_w and N_f from the wall and floor, respectively, as well as gravity acting on the center of mass.

The motion of the center of mass (the center of the ladder) follows the semi-circle (r Sin[θ], r Cos[θ]). The force N_w will accelerate the ladder to the right, while N_f will try and rotate the ladder so that it stays on the wall. However, as θ approaches $\frac{\pi}{2}$, the velocity of the ladder would have to be straight down in order for it to stay attached to the wall and floor, and this would imply that there must be a force acting on it to the left (i.e. N_w would have to be negative) to achieve this. So somewhere before this point $N_w = 0$, and since the normal force can't be negative (unless we set up some rails and force the ladder to stay attached to the wall), past this point the ladder will detaches from the wall. Let's find out when that is.

By conservation of energy,

$$m\,g\,r\,\text{Cos}[\theta_0] = m\,g\,r\,\text{Cos}[\theta] + \frac{1}{2}\,m\left(r\,\dot{\theta}\right)^2 + \frac{1}{2}\,(\eta\,m\,r^2)\,\dot{\theta}^2 \tag{32}$$

where $\dot{\theta}$ appears in the kinetic energy term (because it describes how the center of mass moves) and the same $\dot{\theta}$ appears in the rotational energy term (because θ is the angle between the ladder and the vertical). This allows us to solve for the velocity $v = r\,\dot{\theta}$ as

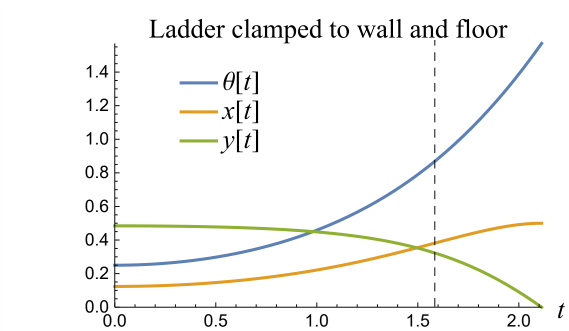
$$\frac{1}{2}\,m\,(1+\eta)\,v^2 = m\,g\,r\,(\text{Cos}[\theta_0] - \text{Cos}[\theta]) \tag{33}$$

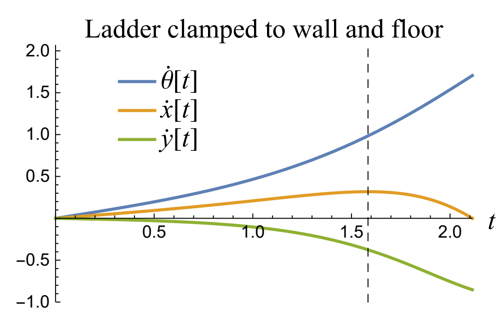
$$v = \left\{\frac{2\,g\,r}{1+\eta}\,(\text{Cos}[\theta_0] - \text{Cos}[\theta])\right\}^{1/2} \tag{34}$$

The velocity has a horizontal component

$$v_x = v\,\text{Cos}[\theta] \propto (\text{Cos}[\theta_0] - \text{Cos}[\theta])^{1/2}\,\text{Cos}[\theta] \tag{35}$$

We want to find out when this is maximized, since a negative slope of v_x (also denoted as $\dot{x}$) implies that N_w points to the left. To clarify this point, here is what the motion of the ladder would look like if the ladder was clamped onto frictionless rails on the floor and wall. The dashed vertical line signifies when Che horizontal velocity $\dot{x}$ reaches its maximum value, at which point $N_w = 0$.





Thus, our goal is to find the time at which $v_x = \dot{x}$ is maximized, assuming that the ladder remains clamped to the floor and wall, which we do by taking the time derivative $\frac{dv_x}{dt}$ and setting it equal to zero. However, a very convenient trick that nearly always works in such situations is to notice that v_x is maximized when v_x^2 is maximized (since v_x is never negative), but this squaring removes the nasty square roots and makes the computation significantly easier. We can also ignore all of the extraneous terms that do not involve θ. Therefore,

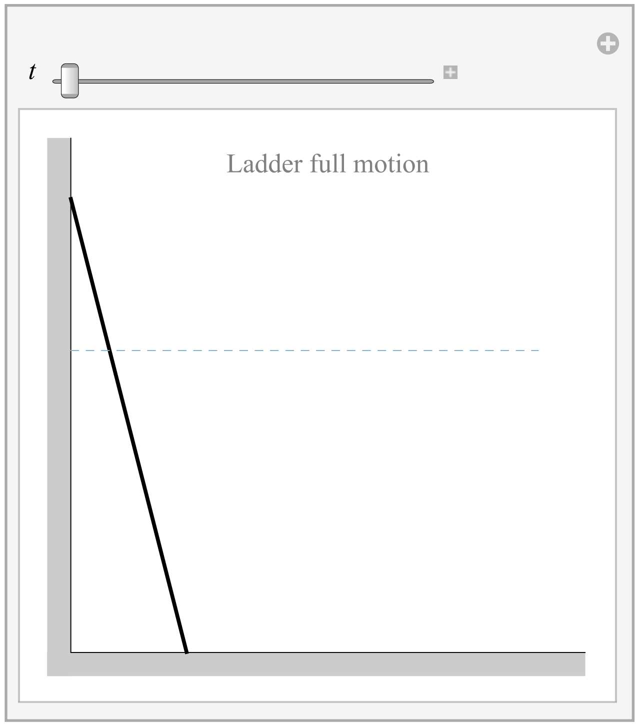
$$\begin{aligned}\frac{dv_x^2}{d\theta} &\propto \text{Sin}[\theta]\,\text{Cos}[\theta]^2 - (\text{Cos}[\theta_0] - \text{Cos}[\theta])\,2\,\text{Cos}[\theta]\,\text{Sin}[\theta] \\ &= 3\,\text{Sin}[\theta]\,\text{Cos}[\theta]^2 - 2\,\text{Cos}[\theta]\,\text{Sin}[\theta]\,\text{Cos}[\theta_0] \\ &= \frac{1}{2}\,\text{Sin}[2\,\theta]\,(3\,\text{Cos}[\theta] - 2\,\text{Cos}[\theta_0])\end{aligned} \tag{36}$$

Aside from the trivial solutions Sin[$2\,\theta$] = 0 when the ladder either begins flat against the floor or against the wall, we see that v_x^2 (and therefore v_x) is maximized when

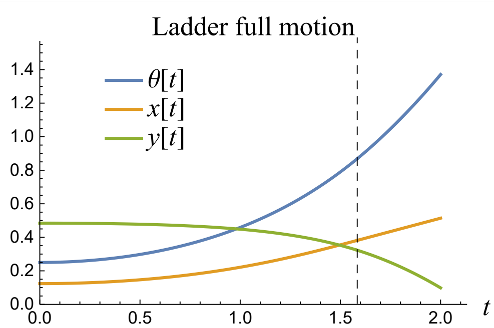
$$\text{Cos}[\theta] = \frac{2}{3}\,\text{Cos}[\theta_0] \tag{37}$$

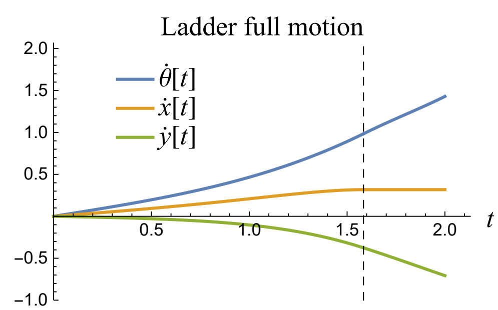
If we multiply both sides of this equation by $2\,r$, we see that the ladder falls when it reaches $\frac{2}{3}$ of its initial height against the wall! Also, this result is independent of η, which implies that for it will hold for any mass distribution, provided that the center of mass is still at the center of the ladder.

The following simulation shows the actual motion of the ladder as it detaches from the wall.



Note that the detachment of the ladder from the wall is very subtle. It is easier to see this change by looking at the plots of $\dot{x}[t]$ over time. When the ladder detaches, $\dot{x}[t]$ = constant because there are no longer any horizontal forces on the ladder.





<u>Solution 2</u>

Until the ladder disconnects from the wall,

$$x = r\,\mathrm{Sin}[\theta] \tag{38}$$

$$y = r\,\mathrm{Cos}[\theta] \tag{39}$$

where we suppress the time dependence of θ. Differentiating twice, we obtain

$$\ddot{x} = -r\,\dot{\theta}^2\,\mathrm{Sin}[\theta] + r\,\ddot{\theta}\,\mathrm{Cos}[\theta] \tag{40}$$

$$\ddot{y} = -r\,\dot{\theta}^2\,\mathrm{Cos}[\theta] - r\,\ddot{\theta}\,\mathrm{Sin}[\theta] \tag{41}$$

Denote the normal forces from the wall and floor as N_w and N_f, respectively. The force equations in the x and y directions and the torque equation (about the center of mass of the ladder) are

$$N_w = m\ddot{x} \tag{42}$$

$$N_f - m\,g = m\ddot{y} \tag{43}$$

$$r\,(N_f \operatorname{Sin}[\theta] - N_w \operatorname{Cos}[\theta]) = \eta\, m\, r^2\, \ddot{\theta} \tag{44}$$

where in the last equation we have used the fact that θ is the angle about which the center of mass rotates. The above 5 equations have 6 unknowns: N_f, N_w, $\ddot{x}$, $\ddot{y}$, $\dot{\theta}$, $\ddot{\theta}$. The last equation comes from conservation of energy

$$m\,g\,r \operatorname{Cos}[\theta_0] = m\,g\,r \operatorname{Cos}[\theta] + \frac{1}{2} m \left(r\,\dot{\theta}\right)^2 + \frac{1}{2}(\eta\, m\, r^2)\,\dot{\theta}^2 \tag{45}$$

```
FullSimplify@Solve[{
    Nw == m x''[t],
    Nf - m g == m y''[t],
    r (Nf * Sin[θ[t]] - Nw * Cos[θ[t]]) == (η m r^2) θ''[t],
    m g r (Cos[θ[0]] - Cos[θ[t]]) == 1/2 m (1 + η) (r θ'[t])^2
   } /. {
    x''[t] → -r (θ'[t])^2 Sin[θ[t]] + r θ''[t] Cos[θ[t]],
    y''[t] → -r (θ'[t])^2 Cos[θ[t]] - r θ''[t] Sin[θ[t]]
   },
  {Nw, Nf, θ'[t], θ''[t]}
 ][[2]]
```

```
{Nw → 1/(1 + η) g m (-2 Cos[θ[0]] + 3 Cos[θ[t]]) Sin[θ[t]], Nf → 1/(2 (1 + η)) g m (3 + 2 η - 4 Cos[θ[0]] Cos[θ[t]] + 3 Cos[2 θ[t]]),
 θ'[t] → (Sqrt[2] Sqrt[g (Cos[θ[0]] - Cos[θ[t]])])/Sqrt[r (1 + η)], θ''[t] → (g Sin[θ[t]])/(r + r η)}
```

Looking at the normal force

$$N_w = \frac{m\,g}{1+\eta} \operatorname{Sin}[\theta]\,(3 \operatorname{Cos}[\theta] - 2 \operatorname{Cos}[\theta_0]) \tag{46}$$

we see that it equals zero for the trivial solutions $\theta = 0$ and $\theta = \frac{\pi}{2}$ (when the ladder either begins flat on the floor or flat against the wall) or the non-trivial solution for a starting position $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ when

$$\operatorname{Cos}[\theta] = \frac{2}{3} \operatorname{Cos}[\theta_0] \tag{47}$$

Notice that although this method was more complicated, we did gain more insight about this problem. In particular, we now have explicit relations for our unknowns N_f, N_w, $\ddot{x}$, $\ddot{y}$, $\dot{\theta}$, and $\ddot{\theta}$ as a function of θ. □

Mathematica Initialization